

A GENERALIZATION OF MARTIN'S AXIOM

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ABSTRACT. We define a certain class Υ of proper posets with the \aleph_2 -chain condition. The corresponding forcing axiom is a generalization of Martin's Axiom; in fact, $\text{MA}(\Upsilon)_{<\kappa}$ implies $\text{MA}_{<\kappa}$. Also, $\text{MA}(\Upsilon)_{<\kappa}$ implies certain uniform failures of club-guessing on ω_1 that don't seem to have been considered in the literature before. We show, assuming CH and given any regular cardinal $\kappa \geq \omega_3$ such that $\mu^{\aleph_1} < \kappa$ for all $\mu < \kappa$ and $\diamond(\{\alpha < \kappa : cf(\alpha) \geq \omega_2\})$ holds, that there is a proper partial order \mathcal{P} of size κ with the \aleph_2 -chain condition and producing a generic extension satisfying $2^{\aleph_0} = \kappa$ together with $\text{MA}(\Upsilon)_{<\kappa}$.

1. A GENERALIZATION OF MARTIN'S AXIOM. AND SOME OF ITS APPLICATIONS.

Martin's Axiom, often denoted by MA, is the following very well-known and very classical forcing axiom: If \mathbb{P} is a partial order (poset, for short) with the countable chain condition¹ and \mathcal{D} is a collection of size less than 2^{\aleph_0} consisting of dense subsets of \mathbb{P} , then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Martin's Axiom is obviously a weakening of the Continuum Hypothesis. Given a cardinal λ , MA_λ is obtained from considering, in the above formulation of MA, collections \mathcal{D} of size at most λ rather than of size less than 2^{\aleph_0} . Martin's Axiom becomes interesting when $2^{\aleph_0} > \aleph_1$.

MA_{ω_1} was the first forcing axiom ever considered ([10]). As observed by D. Martin, the consistency of MA together with $2^{\aleph_0} > \aleph_1$ follows from generalizing the Solovay–Tennenbaum construction of a model of Suslin's Hypothesis by iterated forcing using finite supports ([16]). Since then, a huge number of applications of MA ($+ 2^{\aleph_0} > \aleph_1$) have

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¹A partial order has the countable chain condition if and only if it has no uncountable antichains. More generally, given a cardinal κ , a partial order has the κ -chain condition if it has no antichains of size κ .

been discovered in set theory, topology, measure theory, group theory, and so on ([5] is a classical reference for this).

In the present paper we generalize Martin's Axiom to a certain class of posets Υ with the \aleph_2 -chain condition. In fact, every poset with the countable chain condition will be in Υ , so that for every cardinal λ , the forcing axiom $\text{MA}(\Upsilon)_\lambda$ for Υ relative to collections of size λ of dense sets will imply MA_λ . Furthermore, there will be no restriction on λ other than $\lambda < 2^{\aleph_0}$. More precisely, the same construction will show that $\text{MA}(\Upsilon)_{\aleph_1}$, $\text{MA}(\Upsilon)_{\aleph_{727}}$, $\text{MA}(\Upsilon)_{\aleph_{\omega_2+\omega+3}}$, and so on are all consistent.² This construction will take the form of a forcing iteration, in a broad sense of the expression, involving certain symmetric systems of countable structures as side conditions as in our previous work in [2] and [3].

That Υ cannot possibly consist of all posets with the \aleph_2 -c.c. is clear simply by considering the collapse of ω_1 to ω with finite conditions.³ On the other hand, Υ will be general enough to make the corresponding forcing axiom $\text{MA}(\Upsilon)_\lambda$ strictly stronger than MA_λ . In fact, we will show that $\text{MA}(\Upsilon)_\lambda$ implies certain 'uniform' failures of Club Guessing on ω_1 that don't seem to have been considered before in the literature, and which don't follow from MA_λ . As a matter of fact, we don't know how to show the consistency of these statements by any method other than ours. To be a little more precise, we don't know how to prove their consistency by means of a forcing iteration in the 'conventional' sense.

Let \mathbb{P} be a poset and let N be a sufficiently correct structure such that $\mathbb{P} \in N$. Recall that a \mathbb{P} -condition p is (N, \mathbb{P}) -generic if for every extension p' of p and every dense subset D of \mathbb{P} belonging to N (equivalently, every maximal antichain D of \mathbb{P} belonging to N) there is some condition in $D \cap N$ compatible with p' . Also, \mathbb{P} is proper ([17]) if for every cardinal $\theta \geq |TC(\mathbb{P})|^+$, it holds that for every (equivalently, for club-many) countable $N \prec H(\theta)$ and every $p \in N \cap \mathbb{P}$ there is a condition q in \mathbb{P} extending p and such that q is (N, \mathbb{P}) -generic. Every poset \mathbb{P} with the countable chain condition is proper as every condition is (N, \mathbb{P}) -generic for every N as above.

Now we may proceed to the definition of Υ .

²The same is true for the Solovay-Tennenbaum construction, i.e., the same construction shows the consistency of Martin's Axiom together with 2^{\aleph_0} being \aleph_2 , \aleph_{728} , $\aleph_{\omega_2+\omega+4}$, and so on.

³This poset \mathbb{P} has size \aleph_1 and therefore has the \aleph_2 -c.c. On the other hand, the forcing axiom for collections of \aleph_1 -many dense subsets of \mathbb{P} is obviously false.

Definition 1.1. Given a poset \mathbb{P} , we will say that \mathbb{P} is *regular* if and only if the following holds.

- (a) All its elements are ordered pairs whose first component is a countable ordinal.
- (b) For every regular cardinal $\lambda \geq |TC(\mathbb{P})|^+$ there is a club $D \subseteq [H(\lambda)]^{\aleph_0}$ such that for every finite subset $\{N_i : i \in m\}$ of D and every condition (ν, X) such that $\nu < \min\{N_i \cap \omega_1 : i < m\}$ there is a condition extending (ν, X) and (N_i, P) -generic for all i .⁴

We will say that a poset *admits a regular representation* if it is isomorphic to a regular poset.

Note that every regular poset is proper.

Definition 1.2. Υ is the class of all regular posets with the \aleph_2 -chain condition.

The notation $MA(\Upsilon)_\lambda$ has already shown up.

Definition 1.3. Given a cardinal λ , let $MA(\Upsilon)_\lambda$ be the following statement: For every \mathbb{P} in Υ and for every collection \mathcal{D} of size λ consisting of dense subsets of \mathbb{P} , there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

1.1. Some consequences of $MA(\Upsilon)_\lambda$. Note that every poset \mathbb{P} with the countable chain condition admits a regular representation $\pi : \mathbb{P} \rightarrow \{0\} \times \mathbb{P}$ given by simply setting $\pi(p) = (0, p)$. In particular, for every λ , $MA(\Upsilon)_\lambda$ implies MA_λ .

Also, the following can be proved by arguing very much as in the standard proof that MA_λ implies the productiveness of c.c.c. (see e.g. [9], Lemma 2.23 and Theorem 2.24).

Theorem 1.4. $MA(\Upsilon)_{\aleph_2}$ implies that if $\mathbb{P} \in \Upsilon$ and $X \in [\mathbb{P}]^{\aleph_2}$, then there is $Y \in [X]^{\aleph_2}$ such that every nonempty $\sigma \in [Y]^{<\omega}$ has a lower bound in \mathbb{P} . In particular, any finite support product of members of Υ has the \aleph_2 -chain condition.

Let us see that $MA(\Upsilon)_\lambda$ implies certain uniform failures of Club Guessing on ω_1 . It will be convenient to consider the following natural notion of rank.

Definition 1.5. Given an ordinal α and a set X of ordinals,

- (i) $rank(X, \alpha) > 0$ if and only if α is a limit point of X , and

⁴Note that we are not assuming that $(\nu, X) \in \bigcap \{N_i : i < m\}$.

- (ii) for every ordinal $\eta > 0$, $\text{rank}(X, \alpha) > \eta$ if and only if α if there is $Y \subseteq \alpha$ cofinal in α such that $\text{rank}(X, \beta) \geq \beta$ for all $\beta \in Y$.

Definition 1.6. Given a cardinal λ and a countable ordinal τ , let $(*)_\lambda^\tau$ be the following statement:

For every $\lambda' \leq \lambda$ and every sequence $(A_i)_{i < \lambda'}$, if each A_i is a subset of ω_1 of order type at most τ , then there is a club $C \subseteq \omega_1$ such that $C \cap A_i$ is finite for every $i < \lambda'$.

Definition 1.7. Given a cardinal λ , $(*)_\lambda^+$ is the following statement:

For every $\lambda' \leq \lambda$ and every sequence $(A_i)_{i < \lambda'}$ of infinite subsets of ω_1 there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ and every $i < \lambda'$, if $\text{rank}(A_i, \delta) < \delta$, then $C \cap A_i \cap \delta$ is finite.

Clearly, for every λ , $(*)_\lambda^+$ implies $(*)_\lambda^\tau$ for every $\tau < \omega_1$. Also, if $(\tau, \lambda), (\tau', \lambda')$ are such that $\tau \leq \tau'$ and $\lambda \leq \lambda'$, then $(*)_{\lambda'}^{\tau'}$ implies $(*)_\lambda^\tau$. In particular, for every infinite τ and every $\lambda \geq \omega_1$, $(*)_\lambda^\tau$ implies the negation of Weak Club Guessing on ω_1 .

Fact 1.8. For every cardinal $\lambda \geq \omega_1$, the following weakening of $(*)_\lambda^\omega$ implies $2^{\aleph_0} > \lambda$: For every $\lambda' \leq \lambda$ and every sequence $(A_\delta^i)_{i < \lambda', \delta \in \text{Lim}(\omega_1)}$, if each A_δ^i is a cofinal subset of δ of order type ω , then there is a club $C \subseteq \omega_1$ such that $A_\delta^i \not\subseteq C$ for all $i < \lambda'$ and $\delta \in \text{Lim}(\omega_1)$.

Proof. Suppose $2^{\aleph_0} \leq \lambda$ and let $(A_\delta^i)_{i < \lambda, \delta \in \text{Lim}(\omega_1)}$ be such that for each δ , $\{A_\delta^i : i < \lambda\}$ contains all cofinal subsets of δ . If $C \subseteq \omega_1$ is a club and $\delta \in C$ is a limit point of C , then there is $i < \lambda$ such that $A_\delta^i \subseteq C$. \square

Fact 1.9. For every cardinal $\lambda \geq \omega_1$, $\text{MA}(\Upsilon)_\lambda$ implies $(*)_\lambda^+$.

Proof. Let λ' and $(A_i)_{i < \lambda'}$ be as in the definition of $(*)_\lambda^+$. Let \mathbb{P} consist of all pairs (f, X) such that

- (a) $f \subseteq \omega_1$ is a finite function such that $\text{rank}(f(\nu), f(\nu)) \geq \nu$ for every $\nu \in \text{dom}(f)$,
- (b) X is finite set of triples (i, ν, a) such that $i < \lambda'$, $\nu \in \text{dom}(f)$, $\text{rank}(A_i, f(\nu)) < f(\nu)$, and a is a finite subset of $f(\nu)$, and
- (c) for every $(i, \nu, a) \in X$, $\text{range}(f \upharpoonright \nu) \cap A_i = a$.

Given \mathbb{P} -conditions (f_0, X_0) and (f_1, X_1) , (f_1, X_1) extends (f_0, X_0) if $f_0 \subseteq f_1$ and $X_0 \subseteq X_1$.

It is easy to check that \mathbb{P} admits a regular representation and that it is \aleph_2 -Knaster⁵ (for example by arguments as in [2] for similar forcings). Also, there is a collection \mathcal{D} of $\max\{\lambda', \omega_1\}$ -many dense subsets of ω_1 such that if G is a filter of \mathbb{P} meeting all members of \mathcal{D} , then

⁵Given a cardinal μ , a partial order \mathbb{P} is μ -Knaster if for every $X \in [\mathbb{P}]^\mu$ there is $Y \in [X]^\mu$ consisting of pairwise compatible conditions.

$\text{range}(\bigcup\{f : (f, X) \in G \text{ for some } X\})$ is a club witnessing $(*)_{\lambda}^+$ for $(A_i)_{i < \lambda'}$. \square

On the other hand, no forcing axiom MA_{λ} implies $(*)_{\lambda'}^{\tau}$ for any infinite $\tau < \omega_1$ and any $\lambda' \geq \omega_1$. The reason is simply that MA_{λ} can always be forced by a c.c.c. forcing and c.c.c. forcing preserves Weak Club Guessing.

Definition 1.10. Given a cardinal λ , $(\triangleleft)_{\lambda}$ is the following statement:

Let $\lambda' \leq \lambda$, and suppose $(f_i)_{i < \lambda'}$ is a sequence of functions such that for each i there is some $\alpha_i < \omega_1$ such that $f_i : \alpha_i \rightarrow \omega$ is a continuous function with respect to the order topology. Then there is a club $C \subseteq \omega_1$ such that for all $i < \lambda'$, $\text{range}(f_i \upharpoonright C) \neq \omega$.

$(\triangleleft)_{\lambda}$ clearly implies $\neg\mathfrak{U}$ in J. Moore's terminology ([13]) as well as $2^{\aleph_0} > \lambda$. Also, by the same argument as before, no forcing axiom of the form MA_{λ} implies $(\triangleleft)_{\lambda'}$ for any $\lambda' \geq \omega_1$.

The proof of the following result is as in the proof of Fact 1.9. The proof is similar to the proof that $\text{PFA}^*(\omega_1)_{\omega_1}$ (see below) implies $\neg\mathfrak{U}$ (cf. [2]).

Fact 1.11. For every cardinal λ , $\text{MA}(\Upsilon)_{\lambda}$ implies $(\triangleleft)_{\lambda}$.

Definition 1.12. ([2]) Given a partial order \mathbb{P} , \mathbb{P} is *finitely proper* if for every cardinal $\theta \geq |\mathbb{P}|^+$, every finite sequence $\{N_0, \dots, N_n\}$ of countable elementary substructures of $H(\theta)$ containing \mathbb{P} , and every $p \in N \cap \mathbb{P}$ there is a condition in \mathbb{P} extending p and (N_i, \mathbb{P}) -generic for every $i < n + 1$.

Definition 1.13. ([2], essentially) Given a cardinal λ , $\text{PFA}^*(\omega_1)_{\lambda}$ is the forcing axiom for the class of finitely proper posets of size \aleph_1 and for collections of λ -many dense sets.

Fact 1.14. For every cardinal $\lambda \geq \omega_1$, $\text{PFA}^*(\omega_1)_{\lambda}$ implies the following.

- (1) MA_{ω_1}
- (2) $(*)_{\omega_1}^+$
- (3) $(\triangleleft)_{\omega_1}$
- (4) For every set \mathcal{F} of size λ consisting of functions from ω_1 into ω_1 there is a normal function $g : \omega_1 \rightarrow \omega_1$ such that $\{\nu < \omega_1 : f(\nu) < g(\nu)\}$ is unbounded for every $f \in \mathcal{F}$.

Proof. The proofs of (1)–(3) are either immediate or as the corresponding proofs from $\text{MA}(\Upsilon)_{\lambda}$. (4) follows from considering Baumgartner's forcing for adding a club $C \subseteq \omega_1$ by finite approximations. \square

The following result is straightforward.

Fact 1.15. *Every finitely proper poset of size \aleph_1 is in Υ . In particular, for every cardinal λ , $\text{MA}(\Upsilon)_\lambda$ implies $\text{PFA}^*(\omega_1)_\lambda$.*

The proof of the main theorem in [2] essentially shows the consistency of $\text{PFA}^*(\omega_1)_\lambda$ for arbitrary λ .

2. THE CONSISTENCY OF $\text{MA}(\Upsilon)_\lambda$

Our main theorem is the following:

Theorem 2.1. (CH) *Let $\kappa \geq \omega_3$ be a regular cardinal such that $\mu^{\aleph_1} < \kappa$ for all $\mu < \kappa$ and $\diamond(\{\alpha < \kappa : cf(\alpha) \geq \omega_2\})$ holds. Then there exists a proper forcing notion \mathcal{P} of size κ with the \aleph_2 -chain condition such that the following statements hold in the generic extension by \mathcal{P} :*

- (1) $2^{\aleph_0} = \kappa$
- (2) $\text{MA}(\Upsilon)_{<2^{\aleph_0}}$

The proof of Theorem 2.1 is an elaboration of the proof of the main theorem in [2]. Our approach in that paper consisted in building a certain type of finite support forcing iteration $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ of length κ (in a broad sense of ‘forcing iteration’)⁶ using what one may describe as finite “symmetric” systems of countable elementary substructures of a fixed $H(\kappa)$ ⁷ as side conditions. These systems of structures were added at the first stage \mathcal{P}_0 of the iteration. Roughly speaking, the fact that the supports of the conditions in the iteration was finite ensured that the inductive proofs of the relevant facts – mainly the \aleph_2 -c.c. of all \mathcal{P}_α and their properness – went through. The use of the sets of structures as side conditions was crucial in the proof of properness.⁸ Here we change the set-up from [2] in various ways. One of the changes is the presence of a diamond-sequence which ensures that all proper posets with the \aleph_2 -c.c. (with no restrictions on their size) occurring in the final extension have been dealt with at κ -many stages during the iteration. Of course, Theorem 2.1 shows also that all forcing axioms of the form $\text{MA}(\Upsilon)_{<\kappa}$, for a fixed reasonably defined cardinal κ , are consistent (relative to the consistency of ZFC). As far as we know, these axioms have not been considered in the literature before for $\kappa \geq \aleph_3$.

⁶In the sense that \mathcal{P}_β is a regular extension of \mathcal{P}_α whenever $\alpha < \beta \leq \kappa$. It follows of course that $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is forcing-equivalent to a forcing iteration $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ in the ordinary sense (that is, such that $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ for all α , where $\dot{\mathbb{Q}}_\alpha$ is a \mathbb{P}_α -name for a poset), but such a presentation of $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is not really ‘natural’.

⁷This κ is exactly the value that 2^{\aleph_0} attains at the end of the construction.

⁸For more on the motivation of this type of construction see [2].

Also, Theorem 2.1 shows that no axiom of the form $\text{MA}(\Upsilon)_\lambda$ decides the size of the continuum and thus, by Fact 1.15, fits nicely within the ongoing project of showing whether or not weak fragments of BPFA⁹ imply $2^{\aleph_0} = \aleph_2$. The problem whether (consequences of) forcing axioms for classes of posets with small chain condition decide the size of the continuum does not seem to have received much attention in the literature so far.¹⁰ One place where the problem has been addressed is of course our [2]. Before that, M. Foreman and P. Larson showed in an unpublished note ([6]) that $\text{FA}(\Gamma)$, for Γ being the class of posets of size \aleph_2 preserving stationary subsets of ω_1 , implies $2^{\aleph_0} = \aleph_2$. Several natural problems in this area remain open. For example it is not known whether the forcing axiom for the class of semi-proper posets of size \aleph_2 implies $2^{\aleph_0} = \aleph_2$, and the same is open for the forcing axiom for the class of all posets of size \aleph_1 preserving stationary subsets of ω_1 , and even for the forcing axiom for the class of all proper posets of size \aleph_1 (let us denote these two forcing axioms by, respectively, $\text{MM}(\omega_1)$ and $\text{PFA}(\omega_1)$). It is open whether or not $\text{MM}(\omega_1)$ is equivalent to $\text{PFA}(\omega_1)$,¹¹ and even whether $\text{MM}(\omega_1)$ has consistency strength above ZFC.¹²

As we mentioned before, Theorem 2.1 shows in particular that the forcing axiom $\text{MA}(\Upsilon)_{<\aleph_2}$ has the same consistency strength as ZFC. Other articles dealing with the consistency strength of other (related) fragments of PFA are [11], [7], [15], and [14].

For the most part our notation follows set-theoretic standards as set forth for example in [8] and in [9], but we will also make use of certain *ad hoc* pieces of notation that we introduce now. If N is a set whose intersection with ω_1 is an ordinal, then δ_N will denote this intersection. Throughout this paper, if N and N' are such that there is a (unique) isomorphism from (N, \in) into (N', \in) , then we denote this isomorphism by $\Psi_{N, N'}$. If $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is a forcing iteration and $\alpha \leq \kappa$, \dot{G}_α denotes

⁹BPFA is the assertion that $H(\omega_2)^V$ is a Σ_1 -elementary substructure of $H(\omega_2)^{V^{\mathbb{P}}}$ for every proper poset \mathbb{P} ([4]).

¹⁰Usually the focus has been on deriving $2^{\aleph_0} = \aleph_2$ from bounded forms of forcing axioms, that is forms of forcing axioms in which one considers only small maximal antichains but where the posets are allowed to have large antichains as well.

¹¹On the other hand, $\text{PFA}(\omega_1)$ is trivially equivalent to the forcing axiom for the class of semi-proper posets of size \aleph_1 .

¹²The existence of a non-proper poset of size \aleph_1 preserving stationary subsets of ω_1 is consistent. In fact, Hiroshi Sakai has recently constructed such a poset assuming a suitably strong version of \diamond_{ω_1} holding in L and which can always be forced.

the canonical \mathcal{P}_α -name for the generic filter added by \mathcal{P}_α . Also, \leq_α typically denotes the extension relation on \mathcal{P}_α .

We will make use of the following general fact:

Fact 2.2. *For every $\mathcal{Q} \in \Upsilon$ and every $X \subseteq \mathcal{Q}$ there is \mathcal{R} such that*

- (i) \mathcal{R} is a complete suborder of \mathcal{Q} ,
- (ii) $\mathcal{R} \in \Upsilon$,
- (ii) $X \subseteq \mathcal{R}$, and
- (iii) $|\mathcal{R}| = |X|^{\aleph_1}$

Proof. Let M be an elementary substructure of some large enough $H(\theta)$ containing everything relevant and closed under ω_1 -sequences. Set $\mathcal{R} = \mathcal{Q} \cap M$. Since M is closed under ω_1 -sequences, \mathcal{R} has the \aleph_2 -chain condition and is a complete suborder of \mathcal{Q} .

To see that \mathcal{R} is regular, let $\chi \geq |TC(\mathcal{Q})|^+$ be a cardinal in M and W a well-order of $H(\chi)$ also in M , let $D \subseteq [H(\chi)]^{\aleph_0}$ be a club witnessing the regularity of \mathcal{Q} , let $(\nu, x) \in \mathcal{R}$, and let N_0, \dots, N_m be countable elementary substructures of $(H(\chi), \in, W)$ in D containing \mathcal{Q} and such that $\nu < N_i \cap \omega_1$ for all i . In M , there are countable elementary substructures M_0, \dots, M_m of $(H(\chi), \in, W)$ such that for all i ,

- (a) $M_i \cap N_i = N_i \cap M$,¹³ and
- (b) there is an isomorphism $\varphi_i : (N_i, \in, W) \longrightarrow (M_i, \in, W)$ fixing $N_i \cap M_i$,

and there is a condition $(\bar{\nu}, \bar{x}) \in \mathcal{R}$ such that $(\bar{\nu}, \bar{x}) \leq_{\mathcal{Q}} (\nu, x)$ and such that $(\bar{\nu}, \bar{x})$ is (M_i, \mathcal{Q}) -generic for every i . Suppose towards a contradiction that there is $(\nu', x') \leq_{\mathcal{R}} (\bar{\nu}, \bar{x})$, $i_0 < m + 1$ and some maximal antichain A of \mathcal{R} in N_{i_0} such that no condition in $A \cap N_{i_0}$ is compatible with (ν', x') . Let $(\nu^*, x^*) \in \mathcal{R}$ be a common extension of (ν', x') and of some $(\nu'', x'') \in \varphi_{i_0}(A) \cap M_{i_0}$. To see that there are such (ν^*, x^*) and (ν'', x'') , note that $\varphi_{i_0}(A) \in M_{i_0}$ is a maximal antichain of \mathcal{Q} . This is true since $\mathcal{Q} \in N_{i_0} \cap M_{i_0}$.

Now note that $A \in M$ since M is closed under ω_1 -sequences and $|A| \leq \aleph_1$. It follows that $\varphi_{i_0}(A) = A$ since φ_{i_0} is the identity on $M \cap N_{i_0}$. Also, $(\nu'', x'') \in N_{i_0}$. To see this, take a surjection $f : \omega_1 \longrightarrow A$ in $N_{i_0} \cap M$ (take for example the W -first surjection $f : \omega_1 \longrightarrow A$). Then $\varphi_{i_0}(f) = f \in M_{i_0}$ is a surjection from ω_1 onto $\varphi_{i_0}(A) = A$. Let $\zeta \in M_{i_0} \cap \omega_1$ such that $f(\zeta) = (\nu'', x'')$. Then $\zeta \in N_{i_0} \cap \omega_1$ and so $(\nu'', x'') = f(\zeta) \in N_{i_0}$. This contradiction finishes the proof. \square

¹³In particular, $\mathcal{Q} \in M_{i_0}$.

If $q = (p, \Delta_q)$, where p is a function, we will use $\text{supp}(q)$ to denote $\text{dom}(p)$.¹⁴ If $q = (p, \Delta_q)$ and $r = (s, \Delta_r)$ are ordered pairs, p and s are functions, and α is an ordinal such that $\text{dom}(s) \subseteq \alpha$, then we denote by $q \wedge_\alpha r$ the ordered pair

$$(s \cup (p \upharpoonright_{\text{dom}(p) \setminus \alpha}), \Delta_q \cup \Delta_r)^{15}$$

The rest of the paper is organized as follows: Section 3 contains the construction of our iteration $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ (\mathcal{P}_κ will witness Theorem 2.1). Section 4 contains proofs of the main facts about $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$. Theorem 2.1 follows then easily from these facts.

3. THE FORCING CONSTRUCTION

The proof of Theorem 2.1 will be given in a sequence of lemmas.

Let $\vec{X} = \langle X_\alpha : \alpha \in \kappa, cf(\alpha) \geq \omega_2 \rangle$ be a $\diamond(\{\alpha \in \kappa, cf(\alpha) \geq \omega_2\})$ -sequence. Note that $2^{<\kappa} = \kappa$. Let also Ψ be a well-order of $H(\kappa^+)$ in order type 2^κ . We will define an iteration $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$, together with a function $\Phi : \kappa \rightarrow H(\kappa)$ such that each $\Phi(\alpha)$ is a \mathcal{P}_α -name in $H(\kappa)$.

Let $\langle \theta_\alpha : \alpha \leq \kappa \rangle$ be the strictly increasing sequence of regular cardinals defined as $\theta_0 = |2^\kappa|^+$ and $\theta_\alpha = |2^{\sup\{\theta_\beta : \beta < \alpha\}}|^+$ if $\alpha > 0$. For each $\alpha \leq \kappa$ let \mathcal{M}_α^* be the collection of all countable elementary substructures of $H(\theta_\alpha)$ containing \vec{X} , Ψ , and $\langle \theta_\beta : \beta < \alpha \rangle$. Let also $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathcal{M}_\alpha^*\}$ and note that if $\alpha < \beta$, then \mathcal{M}_α^* belongs to all members of \mathcal{M}_β^* containing the ordinal α .

Our forcing \mathcal{P} will be the direct limit \mathcal{P}_κ of a certain sequence $\langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$ of forcings. The properness of each \mathcal{P}_α will be witnessed by the club \mathcal{M}_α^* . The main idea here is to use the elements of \mathcal{M}_α as side conditions to ensure properness. The actual proof of properness (Lemma 4.7) will be by induction on α . For technical reasons involving the limit case of that proof we need that our side conditions satisfy certain symmetry requirements. These requirements are encapsulated in the following notion of *symmetric system of structures*.

Definition 3.1. Let \mathcal{M} be a club of $[H(\kappa)]^{\aleph_0}$ and let $\{N_i : i < m\}$ be a finite set of members of \mathcal{M} . We will say that $\{N_i : i < m\}$ is a *symmetric system of members of \mathcal{M}* if

- (A) For every $i < m$, $N_i \in \mathcal{M}$.
- (B) Given distinct i, i' in m , if $\delta_{N_i} = \delta_{N_{i'}}$, then there is a (unique) isomorphism

$$\Psi_{N_i, N_{i'}} : (N_i, \in) \longrightarrow (N_{i'}, \in)$$

¹⁴ $\text{supp}(q)$ stands for the *support* of q .

¹⁵Note that the first component of $q \wedge_\alpha r$ is also a function.

Furthermore, we ask that $\Psi_{N_i, N_{i'}}$ be the identity on $\kappa \cap N_i \cap N_{i'}$.

(C) For all i, j in m , if $\delta_{N_j} < \delta_{N_i}$, then there is some $i' < m$ such that $\delta_{N_{i'}} = \delta_{N_i}$ and $N_j \in N_{i'}$.

(D) For all i, i', j in m , if $N_j \in N_i$ and $\delta_{N_i} = \delta_{N_{i'}}$, then there is some $j' < m$ such that $\Psi_{N_i, N_{i'}}(N_j) = N_{j'}$.

If \mathcal{M} is the club of countable $N \preceq H(\kappa)$, we call $\{N_i : i < m\}$ a *symmetric system of elementary substructures of $H(\kappa)$* .

Let us proceed to the definition of $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ now.

Conditions in \mathcal{P}_0 are pairs of the form (\emptyset, Δ) , where

(A) Δ is a finite set of ordered pairs of the form $(N, 0)$ such that $\text{dom}(\Delta)$ is symmetric system of members of \mathcal{M}_0 .

Given \mathcal{P}_0 -conditions $q_\epsilon = (\emptyset, \Delta_\epsilon)$ for $\epsilon \in \{0, 1\}$, q_1 extends q_0 if and only if

(B) $\text{dom}(\Delta_0) \subseteq \text{dom}(\Delta_1)$

Now suppose $\beta \leq \kappa$, $\beta > 0$, and suppose that for each $\alpha < \beta$, \mathcal{P}_α has been defined and is a partial order consisting of pairs of the form $r = (s, \Delta_r)$, where s is a finite function with domain included in α and Δ_r is a set of pairs (N, γ) with $N \in [H(\kappa)]^{\aleph_0}$ and $\gamma \in \alpha + 1$. For each $\alpha < \beta$ let $\mathcal{N}_{\dot{G}_\alpha}$ be a canonical \mathcal{P}_α -name for $\bigcup \{\Delta_r^{-1}(\alpha) : r \in \dot{G}_\alpha\}$.

Definition 3.2. (In $V[G]$ for a \mathcal{P}_α -generic filter G , for a given $\alpha \leq \kappa$ such that \mathcal{P}_α has been defined) A poset $\mathcal{Q} \subseteq H(\kappa)$ is $H(\kappa^+)^V$ -regular relative to G if the following holds:

- (a) All its elements are ordered pairs whose first component is a countable ordinal.
- (b) There is a club D of $[H(\kappa)^V]^{\aleph_0}$ in V such that for every finite set $\{N_i : i \in m\} \subseteq D \cap \mathcal{N}_{\dot{G}_\alpha}$ and every condition $(\nu, X) \in \mathcal{Q}$ with $\nu < \min\{N_i \cap \omega_1 : i < m\}$ there is a condition extending (ν, X) and $(N_i[G], \mathcal{Q})$ -generic for all i .¹⁶

If $\beta = \alpha_0 + 1$, then let $\Phi(\alpha_0)$ be a \mathcal{P}_{α_0} -name in $H(\kappa)$ for an $H(\kappa^+)^V$ -regular poset relative to \dot{G}_{α_0} with the \aleph_2 -chain condition such that $\Phi(\alpha_0)$ is (say) the canonical \mathcal{P}_{α_0} -name for trivial forcing on $\{(0, 0)\}$ unless X_{α_0} is defined and codes (via some fixed reasonable translating function π)¹⁷ a \mathcal{P}_{α_0} -name \dot{X} . In that case, $\Phi(\alpha_0)$ is a \mathcal{P}_{α_0} -name in $H(\kappa)$ for an $H(\kappa^+)^V$ -regular poset relative to \dot{G}_{α_0} with the \aleph_2 -chain

¹⁶Note that every such club D is in $H(\kappa^+)^V$.

¹⁷ π can be taken to be for example the following surjection $\pi : \mathcal{P}(\kappa) \rightarrow H(\kappa^+)$: if $a \in H(\kappa^+)$, then $\pi(X) = a$ if and only if $X \subseteq \kappa$ codes a structure (κ', E) isomorphic to $(TC(\{a\}), \in)$ (for some unique cardinal $\kappa' \leq \kappa$).

condition such that $\Phi(\alpha_0)$ is \dot{X} if \dot{X} is such a poset and such that $\Phi(\alpha_0)$ is (say) trivial forcing on $\{(0, 0)\}$ if \dot{X} is not such a poset.

Let $0 < \beta \leq \kappa$. Conditions in \mathcal{P}_β are pairs of the form $q = (p, \Delta)$ with the following properties.

- (C0) p is a finite function and $\text{dom}(p) \subseteq \beta$.
- (C1) Δ is a finite set of pairs (N, γ) such that $\gamma \leq \beta \cap \sup(N \cap \kappa)$.
- (C2) For every $\alpha < \beta$, the restriction $q|_\alpha$ of q to α is a condition in \mathcal{P}_α , where

$$q|_\alpha := (p \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\})$$

- (C3) If $\alpha \in \text{dom}(p)$, then $p(\alpha) \in H(\kappa)$ and $q|_\alpha \Vdash_{\mathcal{P}_\alpha} p(\alpha) \in \Phi(\alpha)$.
- (C4) If $\alpha \in \text{dom}(p)$, $N \in \mathcal{M}_{\alpha+1}$ and $N \in \Delta_{q|_{\alpha+1}}^{-1}(\alpha + 1)$, then $q|_\alpha$ forces in \mathcal{P}_α that $p(\alpha)$ is $(N[\dot{G}_\alpha], \Phi(\alpha))$ -generic.

Given conditions $q_\epsilon = (p_\epsilon, \Delta_\epsilon)$ (for $\epsilon \in \{0, 1\}$) in \mathcal{P}_β , q_1 extends q_0 if and only if the following holds:

- (D1) $q_1|_\alpha \leq_\alpha q_0|_\alpha$ for all $\alpha < \beta$.
- (D2) $\text{dom}(p_0) \subseteq \text{dom}(p_1)$ and if $\alpha \in \text{dom}(p_0)$, then $q_1|_\alpha$ forces in \mathcal{P}_α that $p_1(\alpha)$ $\Phi(\alpha)$ -extends $p_0(\alpha)$.
- (D3) $\Delta_0^{-1}(\beta) \subseteq \Delta_1^{-1}(\beta)$ if $\beta < \kappa$.

4. THE MAIN FACTS

We are going to prove the relevant properties of the forcings \mathcal{P}_α . Theorem 2.1 will follow immediately from them.

Our first lemma is immediate from the definitions.

Lemma 4.1. $\mathcal{P}_\kappa = \bigcup_{\beta < \kappa} \mathcal{P}_\beta$, and $\emptyset \neq \mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ for all $\alpha \leq \beta \leq \kappa$.

Lemma 4.2 shows in particular that $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is a forcing iteration in a broad sense.

Lemma 4.2. Let $\beta \leq \kappa$ be an ordinal. If $\alpha < \beta \leq \kappa$, $r \in \mathcal{P}_\alpha$, $q \in \mathcal{P}_\beta$, and $r \leq_\alpha q|_\alpha$, then $q \wedge_\alpha r$ is a condition in \mathcal{P}_β extending q . In particular, \mathcal{P}_α is a complete suborder of \mathcal{P}_β .

Proof. This proof makes use of the fact that models in $\text{dom}(\Delta_{q \wedge_\alpha r})$ come always equipped with suitable markers γ . New side conditions (N, γ) appearing in Δ_r may well have the property that $[\alpha, \beta] \cap N \neq \emptyset$, but they will not impose any problematic restraints – coming from clause (C4) in the definition – on any $p(\xi)$ for $\xi \in [\alpha, \beta]$. The reason is simply that $\gamma \leq \alpha$. □

The following lemma gives a representation of $\mathcal{P}_{\alpha+1}$ as a certain dense subset of an iteration of the form $\mathcal{P}_\alpha * \dot{\mathcal{Q}}_\alpha$.

Lemma 4.3. *For all $\alpha < \kappa$, $\mathcal{P}_{\alpha+1}$ is isomorphic to a dense suborder of $\mathcal{P}_\alpha * \dot{\mathcal{Q}}_\alpha$, where $\dot{\mathcal{Q}}_\alpha$ is, in $V^{\mathcal{P}_\alpha}$, the collection of all pairs (v, \mathcal{Q}) such that*

(o) *there is some $r = (p, \Delta) \in \dot{G}_\alpha$ such that*

$$(p \cup \{\langle \alpha, v \rangle\}, \Delta \cup \{(N, \alpha + 1) : N \in \mathcal{Q}\}) \in \mathcal{P}_{\alpha+1},^{18}$$

ordered by $(v_1, \mathcal{Q}_1) \leq_{\dot{\mathcal{Q}}_\alpha} (v_0, \mathcal{Q}_0)$ if and only if

- (i) $\mathcal{Q}_0 \subseteq \mathcal{Q}_1$, and
- (ii) $v_1 \Phi(\alpha)$ -extends v_0 .

Proof. Let $\tilde{\mathcal{P}}_{\alpha+1}$ consist of all (r, \check{x}) , where $r \in \mathcal{P}_\alpha$ and $r \Vdash_{\mathcal{P}_\alpha} \check{x} \in \dot{\mathcal{Q}}_\alpha$. Then $\psi : \mathcal{P}_{\alpha+1} \longrightarrow \tilde{\mathcal{P}}_{\alpha+1}$ is an isomorphism, where $\psi(q) = (q|_\alpha, \check{x})$ for $x = (v, \Delta^{-1}(\alpha + 1))$ if $q = (p \cup \{\langle \alpha, v \rangle\}, \Delta)$. \square

The next step in the proof of Theorem 2.1 will be to show that all \mathcal{P}_α (for $\alpha \leq \kappa$) have the \aleph_2 -chain condition.

Lemma 4.4. *For every ordinal $\beta \leq \kappa$, \mathcal{P}_β has the \aleph_2 -chain condition.*

Proof. The proof is by induction on β . The conclusion for $\beta = 0$ holds by a simplified version of the Δ -system argument (using CH) we will see in a moment in the limit case.

For $\beta = \alpha + 1$ the conclusion follows immediately from Lemma 4.3 together with the induction hypothesis for α , the fact that the poset $\dot{\mathcal{Q}}_\alpha$ in Lemma 4.3 is forced by \mathcal{P}_α to have the \aleph_2 -c.c. (since this is true for $\Phi(\alpha)$), and the fact that the \aleph_2 -c.c. is preserved under forcing iterations of length 2.

Now suppose $\beta \leq \kappa$ is a nonzero limit ordinal. Let q_ξ be \mathcal{P}_β -conditions for $\xi < \omega_2$. By a Δ -system argument using CH we may assume that $\text{dom}(\Delta_{q_\xi} \cup \Delta_{q_{\xi'}})$ is a symmetric system of structures, that $\{\text{supp}(q_\xi) : \xi < \omega_2\}$ forms a Δ -system with root R and, furthermore, that for all distinct ξ, ξ' in ω_2 , $\text{supp}(q_\xi) \setminus R$ has empty intersection with $\bigcup \text{dom}(\Delta_{q_{\xi'}})$. Let $\alpha < \beta$ be a bound for R . By induction hypothesis we may find distinct ξ, ξ' such that $q_\xi|_\alpha$ and $q_{\xi'}|_\alpha$ are compatible \mathcal{P}_α -conditions. Let r be a common extension of $q_\xi|_\alpha$ and $q_{\xi'}|_\alpha$. It follows now that the natural amalgamation of r , q_ξ and $q_{\xi'}$ is a common extension of q_ξ and $q_{\xi'}$. The case $\beta = \kappa$ follows also from $\mathcal{P}_\kappa = \bigcup_{\beta < \kappa} \mathcal{P}_\beta$ together with $\text{cf}(\kappa) \geq \omega_3$. \square

Corollary 4.5. *For every $\beta \leq \kappa$, \mathcal{P}_β forces $H(\kappa)^{V[\dot{G}_\beta]} = H(\kappa)^V[\dot{G}_\beta]$ and forces $N^*[\dot{G}_\beta] \cap H(\kappa) = (N^* \cap H(\kappa))[\dot{G}_\beta]$ whenever $\theta \geq \kappa^+$ is*

¹⁸Note that v is an ordered pair whose first component is a countable ordinal.

regular and N^* is a countable elementary substructure of $H(\theta)$ such that $\mathcal{P}_\beta \in N^*$.

Definition 4.6. Given $\alpha \leq \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a countable elementary substructure $N \prec H(\kappa)$, we will say that q is (N, \mathcal{P}_α) -pre-generic in case

- (\circ) $\alpha < \kappa$ and the pair (N, α) is in Δ_q , or else
- (\circ) $\alpha = \kappa$ and the pair $(N, \sup(N \cap \kappa))$ is in Δ_q .

The properness of all \mathcal{P}_α is an immediate consequence of the following lemma.

Lemma 4.7. Suppose $\alpha \leq \kappa$ and $N^* \in \mathcal{M}_\alpha^*$. Let $N = N^* \cap H(\kappa)$. Then the following conditions hold.

- (1) $_\alpha$ For every $q \in N$ there is $q' \leq_\alpha q$ such that q' is (N, \mathcal{P}_α) -pre-generic.
- (2) $_\alpha$ If $\mathcal{P}_\alpha \in N^*$ and $q \in \mathcal{P}_\alpha$ is (N, \mathcal{P}_α) -pre-generic, then q is $(N^*, \mathcal{P}_\alpha)$ -generic.

Proof. The proof will be by induction on α . We start with the case $\alpha = 0$. For simplicity we are going to identify a \mathcal{P}_0 -condition $q = (\emptyset, \Delta_q)$ with $\text{dom}(\Delta_q)$. The proof of (1) $_0$ is trivial: It suffices to set $q' = q \cup \{(N, 0)\}$.

The proof of (2) $_0$ is also easy: Let E be a dense subset of \mathcal{P}_0 in N^* . It suffices to show that there is some condition in $E \cap N^*$ compatible with q . Notice that $q \cap N^* \in \mathcal{P}_0$. Hence, we may find a condition $q^\circ \in E \cap N^*$ extending $q \cap N^*$. Now let

$$q^* = q \cup \{\Psi_{N, \overline{N}}(M) : M \in q^\circ, \overline{N} \in \text{dom}(\Delta_q), \delta_{\overline{N}} = \delta_N\}$$

It takes a routine verification to check that q^* is a condition in \mathcal{P}_0 extending both q and q° .

Let us proceed to the more substantial case $\alpha = \sigma + 1$. We start by proving (1) $_\alpha$. Without loss of generality we may assume that σ is in the support of q (otherwise the proof is easier). So, q is of the form $q = (p \cup \{(\sigma, v)\}, \Delta_q)$, where v is an ordered pair whose first component is a countable ordinal less than δ_N . By (1) $_\sigma$ we may assume that there is a condition $t \in \mathcal{P}_\sigma$ extending $q|_\sigma$ and (N, \mathcal{P}_σ) -pre-generic. In $V^{\mathcal{P}_\sigma \restriction t}$ let \dot{D} be the Ψ -first club in $V \cap N^*[\dot{G}_\sigma]$ witnessing the $H(\kappa^+)^V$ -regularity of $\Phi(\sigma)$ relative to \dot{G}_σ ,¹⁹ and note that $N \in \dot{D}$ since \dot{D} is closed in V and $N[\dot{G}_\sigma] \cap V = N$. There is then some $v^* \in H(\kappa)$ which is a

¹⁹We can find such a club in $N^*[\dot{G}_\sigma]$ since $\Psi \in N^*$, $N^*[\dot{G}_\sigma] \prec H((2^\kappa)^+)[\dot{G}_\sigma]$, and $\mathcal{P}^V(H(\kappa)^V) \in N^*[\dot{G}_\sigma]$.

$(N[\dot{G}_\sigma], \Phi(\sigma))$ -generic condition $\Phi(\sigma)$ -extending v . Now it suffices to pick $a = (b, \Delta_a) \leq_\sigma$ -extending t and deciding v^* and set

$$q^\dagger = (b \cup \{\langle \sigma, v^* \rangle\}, \Delta_a \cup \Delta_q \cup \{(N, \alpha)\})$$

Remark 4.8. Starting from $\sigma \notin \text{supp}(q)$, we can also run the same argument and find a condition q' extending q and such that $\sigma \in \text{supp}(q)$.

Proof. This is true since $(2)_\sigma$ guarantees that $q|_\sigma$ is also $(M^*, \mathcal{P}_\sigma)$ -generic for all $M \in \Delta_q^{-1}(\sigma+1) \cap \mathcal{M}_{\sigma+1}$ (which implies that the above t forces that all these M are in \dot{D}). Also note that, by its being definable, the first component ν of the weakest condition of $\Phi(\sigma)$ is such that $\nu < \delta_M = \delta_{M^*[\dot{G}_\sigma]}$ for all these M 's. \square

Now let us prove $(2)_\alpha$. Let A be a maximal antichain of \mathcal{P}_α in N^* , and assume without loss of generality that $q = (p, \Delta_q)$ extends some condition q^* in A . We must show $q^* \in N$. Note that A is in N by the \aleph_2 -c.c. of \mathcal{P}_α . Let us work in $V^{\mathcal{P}_\sigma \restriction (q|_\sigma)}$. Let E be the set of $\Phi(\sigma)$ -conditions v such that either

- (i) there exists some $a = (b, \Delta_a) \in \mathcal{P}_\alpha$ extending some member of A such that $a|_\sigma \in \dot{G}_\sigma$, $\sigma \in \text{dom}(b)$, and such that $b(\sigma) = v$, or else
- (ii) there is no $a = (b, \Delta_a) \in \mathcal{P}_\alpha$ extending any member of A such that $a|_\sigma \in \dot{G}_\sigma$, $\sigma \in \text{dom}(b)$, and such that $b(\sigma) \leq_{\Phi(\sigma)} v$.

E is a dense subset of $\Phi(\sigma)$, and $E \in N^*[\dot{G}_\sigma]$ since $N^*[\dot{G}_\sigma] \preceq H(\kappa^+)^V[\dot{G}_\sigma]$ and N^* contains \mathcal{P}_α . Note that E is in fact in $N[\dot{G}_\sigma]$ by Corollary 4.5. Suppose $\sigma \in \text{dom}(p)$ and suppose $p(\sigma) = \bar{v}$ (the case $\sigma \notin \text{dom}(p)$ is slightly simpler). Since \bar{v} is $(N[\dot{G}_\sigma], \Phi(\sigma))$ -generic, we may find some $v' \in E \cap N[\dot{G}_\sigma]$ and some v^* $\Phi(\sigma)$ -extending both v' and \bar{v} .

Claim 4.9. *Condition (i) above holds for v' .*

Proof. Let $r = (s, \Delta_r)$ be a condition in \dot{G}_σ extending $q|_\sigma$ and deciding v^* , and let $u = (s \cup \{\langle \sigma, v^* \rangle\}, \Delta_r \cup \Delta_q)$. Note that u is a \mathcal{P}_α -condition extending q . In particular, u extends a condition in A , and therefore it witnesses the negation of condition (ii) for ν' , so condition (i) must hold for ν' . \square

By the above claim and by $N^*[G_\sigma] \preceq H(\kappa^+)^V[G_\sigma]$ there is $a = (b, \Delta_a)$ in $N^*[G_\sigma]$ witnessing that condition (i) holds for v' , and actually $a \in N$ since $N^*[G_\sigma] \cap V = N^*$ by induction hypothesis. Now we extend $q|_\sigma$ to a condition $r = (s, \Delta_r)$ deciding a , and deciding also some common

extension $v^* \in \Phi(\sigma)$ of \bar{v} and v' . We may also assume that r extends $a|_\sigma$. Now it is straightforward to verify, by the usual arguments, that $(s \cup \{\langle \sigma, v^* \rangle\}, \Delta_r \cup \Delta_a \cup \Delta_q)$ is a \mathcal{P}_α -condition extending a and q . It follows that $q^* = a$.

It remains to prove the lemma for the case when α is a nonzero limit ordinal. The proof of $(1)_\alpha$ is easy. Let $\sigma \in N \cap \alpha$ be above $\text{supp}(q)$. By induction hypothesis we may find $r \in \mathcal{P}_\sigma$ extending $q|_\sigma$ and such that $(N, \sigma) \in \Delta_r$. Check that the result of stretching the marker σ in (N, σ) up to α if $\alpha < \kappa$ and up to $\text{sup}(N \cap \kappa)$ if $\alpha = \kappa$ is a condition in \mathcal{P}_α extending q with the desired property.

For $(2)_\alpha$, let A be a maximal antichain of \mathcal{P}_α in N^* , and assume without loss of generality that $q = (p, \Delta_q)$ extends some condition q^* in A . We must show $q^* \in N$. Suppose first that $\text{cf}(\alpha) = \omega$. In this case we may take $\sigma \in N^* \cap \alpha$ above $\text{supp}(q)$. Let G_σ be \mathcal{P}_σ -generic with $q|_\sigma \in G_\sigma$. In $N^*[G_\sigma]$ it is true that there is a condition $q^\circ \in \mathcal{P}_\alpha$ such that

- (a) $q^\circ \in A$ and $q^\circ|_\sigma \in G_\sigma$, and
- (b) $\text{supp}(q^\circ) \subseteq \sigma$.

(the existence of such a q° is witnessed in $V[G_\sigma]$ by q^* .)

Since $q|_\sigma$ is $(N^*, \mathcal{P}_\sigma)$ -generic by induction hypothesis, $q^\circ \in N^*$. By extending q below σ if necessary, we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. But now, if $q = (p, \Delta_q)$, the natural amalgamation $(p, \Delta_q \cup \Delta_{q^\circ})$ of q and q° is a \mathcal{P}_α -condition extending them. It follows that $q^* = q^\circ$.

Finally, suppose $\text{cf}(\alpha) \geq \omega_1$. This will be the only place where we use the symmetry of $\text{dom}(\Delta)$ for every \mathcal{P}_α condition (p', Δ) . Notice that if $N' \in \text{dom}(\Delta_q)$ and $\delta_{N'} < \delta_N$, then $\text{sup}(N' \cap N \cap \alpha) \leq \text{sup}(\Psi_{\bar{N}, N}(N') \cap \alpha) \in N \cap \alpha^{20}$ whenever $\bar{N} \in \text{dom}(\Delta_q)$ is such that $\delta_{\bar{N}} = \delta_N$ and $N' \in \bar{N}$. Hence we may fix $\sigma \in N \cap \alpha$ above $\text{supp}(q) \cap N$ and above $\text{sup}(N' \cap N \cap \alpha)$ for all $N' \in \text{dom}(\Delta_q)$ with $\delta_{N'} < \delta_N$.

As in the above case, if G_σ is \mathcal{P}_σ -generic with $q|_\sigma \in G_\sigma$, then in $N^*[G_\sigma]$ we can find a condition $q^\circ = (p^\circ, \Delta_{q^\circ}) \in \mathcal{P}_\alpha$ such that $q^\circ \in A$ and $q^\circ|_\sigma \in G_\sigma$ (again, the existence of such a condition is witnessed in $V[G_\sigma]$ by q), and such a q° will necessarily be in N^* . By extending q below σ we may assume that $q|_\sigma$ decides q° and extends $q^\circ|_\sigma$. The proof of $(2)_\alpha$ in this case will be finished if we can show that there is a condition q^\dagger extending q and q° . The condition q^\dagger can be built by recursion on $\text{supp}(q) \cup \text{supp}(q^\circ)$. This finite construction mimics the proof of $(1)_\alpha$ for successor α . Note for instance that if η is in

²⁰Recall that $\Psi_{\bar{N}, N}$ fixes $\bar{N} \cap N \cap \kappa$.

the support of q° and $\sigma \leq \eta < \alpha$, then $p^\circ(\eta) = v$ is an ordered pair whose first component is a countable ordinal less than δ_N . Such an η satisfies that if $\eta \in N'$ for some $N' \in \text{dom}(\Delta_q)$, then $\delta_{N'} \geq \delta_N$. Hence, there must exist an ordered pair v^* and a common extension of $q|_\eta$ and $q^\circ|_\eta$ forcing that $v^* \Phi(\eta)$ -extends v and is $(N'[\dot{G}_\eta], \Phi(\eta))$ -generic for all relevant N' . The reason is that there is, in $V^{\mathcal{P}_\eta \upharpoonright (q^\dagger|_\eta)}$, a club \dot{D} witnessing the $H(\kappa^+)^V$ -regularity of $\Phi(\eta)$ relative to \dot{G}_η , and such that every relevant N' is in \dot{D} . This \dot{D} can be taken to be the first club, in the well-order of $H(\kappa^+)[\dot{G}_\eta]$ induced by Ψ , witnessing the $H(\kappa^+)^V$ -regularity of $\Phi(\eta)$ relative to \dot{G}_η (it is clear that, since all relevant N' contain Ψ , they contain a name for \dot{D} , and therefore are in \dot{D} by $(2)_\eta$). This finishes the proof of $(2)_\alpha$ for limit α and the proof of the lemma. \square

Corollary 4.10. *For all $\alpha \leq \kappa$, \mathcal{P}_α is proper.*

The following two lemmas are trivial.

Lemma 4.11. *For every $\alpha < \kappa$ and every condition $q \in \mathcal{P}_\kappa$, q forces that the collection of all v such that there is some $(p, \Delta) \in \dot{G}_\kappa$ with $p(\alpha) = v$ generates a $V[\dot{G}_\alpha]$ -generic filter on $\Phi(\alpha)$.*

Proof. See Remark 4.8. \square

Lemma 4.12. \mathcal{P}_κ forces that every regular poset $\mathcal{R} \subseteq H(\kappa)$ is $H(\kappa^+)^V$ -regular relative to \dot{G}_κ .

Lemma 4.13 follows from the usual counting of nice names for subsets of κ using $(\kappa^{<\kappa})^V = \kappa$ and Lemma 4.4.

Lemma 4.13. \mathcal{P}_κ forces $\kappa^{<\kappa} = \kappa$.

Lemma 4.15 will make use of the following result.

Lemma 4.14. *Let Q be an elementary substructure of $H(\theta)$, for some large enough θ , and suppose Q is closed under ω_1 -sequences and contains Ψ and \vec{X} . Suppose $Q \cap \kappa$ is an ordinal δ in κ . Then for every \mathcal{P}_δ -condition q there is a \mathcal{P}_δ -condition $q^* \in Q$ such that every condition \leq_δ -extending q is compatible with q^* and every condition \leq_δ -extending q^* is compatible with q .*

Proof. Suppose $q = (p, \Delta_q)$ and $\Delta_q = \{(N_i, \gamma_i) : i < n\}$. For all i let $\tilde{\gamma}_i = \gamma_i$ if $\gamma_i < \delta$, and let $\tilde{\gamma}_i = \sup(N_i \cap \delta)$ if $\gamma_i = \delta$. Note that p is in Q since $X_\alpha \in Q$ for each $\alpha < \delta$. Since $Q \preceq H(\theta)$ contains all reals and $\{(N_i \cap Q, \tilde{\gamma}_i) : i < n\} \in Q$, we may find in Q a set $\{M_i : i < n\}$ with the property that for all $i \in n$ and $\alpha \in N_i \cap Q$,

$\tilde{\gamma}_i \cap N_i = \tilde{\gamma}_i \cap M_i$, $M_i \cap N_i = N_i \cap Q$, $M_i \in \mathcal{M}_{\alpha+1}$ iff $N_i \in \mathcal{M}_{\alpha+1}$, and there is an isomorphism $\varphi_i : (M_i, \in) \longrightarrow (N_i, \in)$ fixing $N_i \cap Q$, and such that $q^* = (p, \{(M_i, \tilde{\gamma}_i) : i \in n\})$ is in \mathcal{P}_κ .

Let $q' = (p', \Delta')$ be a condition in \mathcal{P}_δ extending q . We want to show that $\tilde{q} = (p', \Delta' \cup \Delta_{q^*})$ is a \mathcal{P}_δ -condition (the proof that every condition in \mathcal{P}_δ extending q^* is compatible with q is similar). We prove by induction on $\alpha \leq \delta$ that $\tilde{q}|_\alpha$ is a condition in \mathcal{P}_α . Let $\alpha < \delta$ and suppose $\tilde{q}|_\alpha \in \mathcal{P}_\alpha$. It suffices to show that if $\alpha \in \text{dom}(p')$, then $\tilde{q}|_\alpha$ forces that $p'(\alpha)$ is $(M_i[\dot{G}_\alpha], \Phi(\alpha))$ -generic for every $i < n$ such that $\tilde{\gamma}_i > \alpha$ and such that $M_i \in \mathcal{M}_{\alpha+1}$.

Note that $N_i \in \mathcal{M}_{\alpha+1}$ and that $\gamma_i > \alpha$. Hence, $\tilde{q}|_\alpha$ forces that $p'(\alpha)$ is $(N_i[\dot{G}_\alpha], \Phi(\alpha))$ -generic. Work now in $V^{\mathcal{P}_\alpha}(\tilde{q}|_\alpha)$ and suppose towards a contradiction that there is a condition $y \leq_{\Phi(\alpha)} p'(\alpha)$ and a maximal antichain A of $\Phi(\alpha)$ in $M_i[\dot{G}_\alpha]$ such that no condition in $A \cap M_i[\dot{G}_\alpha]$ is compatible with y . Let $\dot{A} \in M_i$ be a \mathcal{P}_α -name for A . Then $\varphi_i(\dot{A}) \in N_i$ is a \mathcal{P}_α -name for a maximal antichain of $\Phi(\alpha)$ (note that both \mathcal{P}_α and $\Phi(\alpha)$ are fixed by the isomorphism φ_i since these objects are in $N_i \cap Q$) and, by the \aleph_2 -c.c. of \mathcal{P}_α together with the \aleph_2 -c.c. of $\Phi(\alpha)$ in $V^{\mathcal{P}_\alpha}$ and the closure of Q under ω_1 -sequences, $\varphi_i(\dot{A}) \in Q$. It follows that $\varphi_i(\dot{A}) = \dot{A}$. The rest of the argument is as in the proof of Fact 2.2, using the fact that there is a surjection from ω_1 onto \dot{A} in N_i fixed by φ_i^{-1} and the fact that $N_i[\dot{G}_\alpha] \cap \omega_1 = N_i \cap \omega_1 = M_i \cap \omega_1 = M_i[\dot{G}_\alpha] \cap \omega_1$ is forced by $\tilde{q}|_\alpha$ thanks to Lemma 4.7. \square

Given Q and q as in the hypothesis of Lemma 4.14, we will say that the condition q^* given by its conclusion is a *projection of q to Q* .

Lemma 4.15. \mathcal{P}_κ forces $\text{MA}(\Upsilon)_{<\kappa}$.

Proof. Let q be a \mathcal{P}_κ -condition, let $\chi < \kappa$, and let $\dot{\mathcal{R}}$ and \dot{A}_i ($i < \chi$) be \mathcal{P}_κ -names such that q forces that $\dot{\mathcal{R}}$ is an \aleph_2 -c.c. $H(\kappa^+)^V$ -regular poset relative to \dot{G}_κ defined on κ and that each \dot{A}_i is a maximal antichain of $\dot{\mathcal{R}}$. By Lemma 4.13, Fact 2.2, and Lemma 4.12 it suffices to show that there is some condition extending q and forcing that there is a filter on $\dot{\mathcal{R}}$ intersecting all \dot{A}_i . Let X be a subset of κ coding the \mathcal{P}_κ -name $\dot{\mathcal{R}}$ via our fixed translating function π .

Now, using the fact that \vec{X} is a $\diamond(\{\alpha < \kappa : cf(\alpha) \geq \omega_2\})$ -sequence we may fix an elementary substructure Q of some large enough $H(\theta)$ containing q , \mathcal{P}_κ , $\dot{\mathcal{R}}$, $(\dot{A}_i)_{i \in \chi}$, \vec{X} and X , closed under ω_1 -sequences, and such that $\delta = Q \cap \kappa$ is an ordinal such that $X_\delta = X \cap \delta$ (since $\mu^{\aleph_1} < \kappa$ for all $\mu < \kappa$, the set of $\delta \in \kappa$ for which there is a Q as above contains a λ -club for every regular cardinal $\lambda < \kappa$, $\lambda \geq \omega_2$). Furthermore we

may assume that q forces for all ξ, ξ' in δ that if $\pi(\xi)$ and $\pi(\xi')$ are compatible conditions in $\dot{\mathcal{R}}$, then there is an ordinal below δ coding a common extension in $\dot{\mathcal{R}}$ of $\pi(\xi)$ and $\pi(\xi')$.

The following claim follows from the closure of Q under ω_1 -sequences together with the above choice of δ .

Claim 4.16. *Letting $\dot{\mathcal{R}}_0$ be the \mathcal{P}_δ -name coded by X_δ , $\mathcal{P}_\delta \restriction q$ forces that $\dot{\mathcal{R}}_0$ is an $H(\kappa^+)^V$ -regular poset relative to \dot{G}_δ with the \aleph_2 -c.c.*

Proof. The \aleph_2 -c.c. of $\dot{\mathcal{R}}_0$ in $V^{\mathcal{P}_\delta \restriction q}$ follows from Lemma 4.14 together with the fact that q forces for all ξ, ξ' in δ that if ξ and ξ' code compatible conditions \dot{c}, \dot{c}' in $\dot{\mathcal{R}}$, then there is an ordinal below δ coding a common extension in $\dot{\mathcal{R}}$ of \dot{c} and \dot{c}' . In fact, since Q is closed under ω_1 -sequences, q forces $\dot{\mathcal{R}}_0$ to be a complete suborder of $\dot{\mathcal{R}}$. The proof of the $H(\kappa^+)^V$ -regularity of $\dot{\mathcal{R}}_0$ in $V^{\mathcal{P}_\delta \restriction q}$ relative to \dot{G}_δ is essentially as in the proof of Lemma 4.14. \square

It follows from the above claim that q forces $\Phi(\delta) = \dot{\mathcal{R}}_0$. Finally, we may extend q to a condition q' such that $\delta \in \text{supp}(q')$. Then, by Lemma 4.11, q' forces that there is a filter H on $\dot{\mathcal{R}}_0$ meeting all \dot{A}_i , and of course H generates a filter on $\dot{\mathcal{R}}$. \square

Lemma 4.17. \mathcal{P}_κ forces $2^{\aleph_0} = \kappa$.

Proof. $V^{\mathcal{P}_\kappa} \models 2^{\aleph_0} \geq \kappa$ follows for example from the fact that \mathcal{P}_κ forces $\text{MA}(\Upsilon)_{<\kappa}$. $V^{\mathcal{P}_\kappa} \models 2^{\aleph_0} \leq \kappa$ follows from Lemma 4.13. \square

Lemma 4.17 finishes the proof of Theorem 2.1.

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